Proof of Field D*'s Case Separation for Arbitrary Simplices

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Abstract

In their development of the Field D* algorithm [2], the authors prove that a path through a unit length right-angled triangle originating from an interpolated edge, and travelling to the opposite vertex must either be a *direct* or *indirect* case. A combination of the two is not optimal. Later work [5] proves this for arbitrary, but non-degenerate triangles.

In this technical report, we prove the same for *non-degenerate simplices*, which are generalisations of triangles to higher dimensions.

1 Introduction

Common path-finding algorithms such as Dijkstra's shortest path [1] and A* [3, 4] find shortest paths through a graph composed of nodes, and edges weighted with some traversal cost. The shortest path is calculated between a start node and a goal node, and can be defined as the set of edges with minimal summed traversal cost. During the course of execution, these algorithms store the *path cost* at each visited node of the graph. This path cost, denoted by $g(s)$ represents the shortest path cost to the visited node s.

Field D* [2] is a path-finding algorithm that finds shortest paths through the weighted square cells of a grid. The shortest path produced by this algorithm is not constrained to the edges of the grid cells, and may pass through them. Similarly to the aforementioned graph-based algorithms, Field D* stores the shortest path cost at each visited node. While graph-based algorithms can simply add a neighbouring node's path cost to an edge's traversal cost to produce the shortest path cost at a node, Field D* must consider a continuous range of paths through a neighbouring cell and for that reason must minimise *cost functions* when calculating the shortest path to a node.

Field D*'s *general cost function* expresses the cost of a path travelling through one right-angled unit triangle of a square $AB_1B_2B_3$. It quantifies the cost of a path, shown in Figure 1a

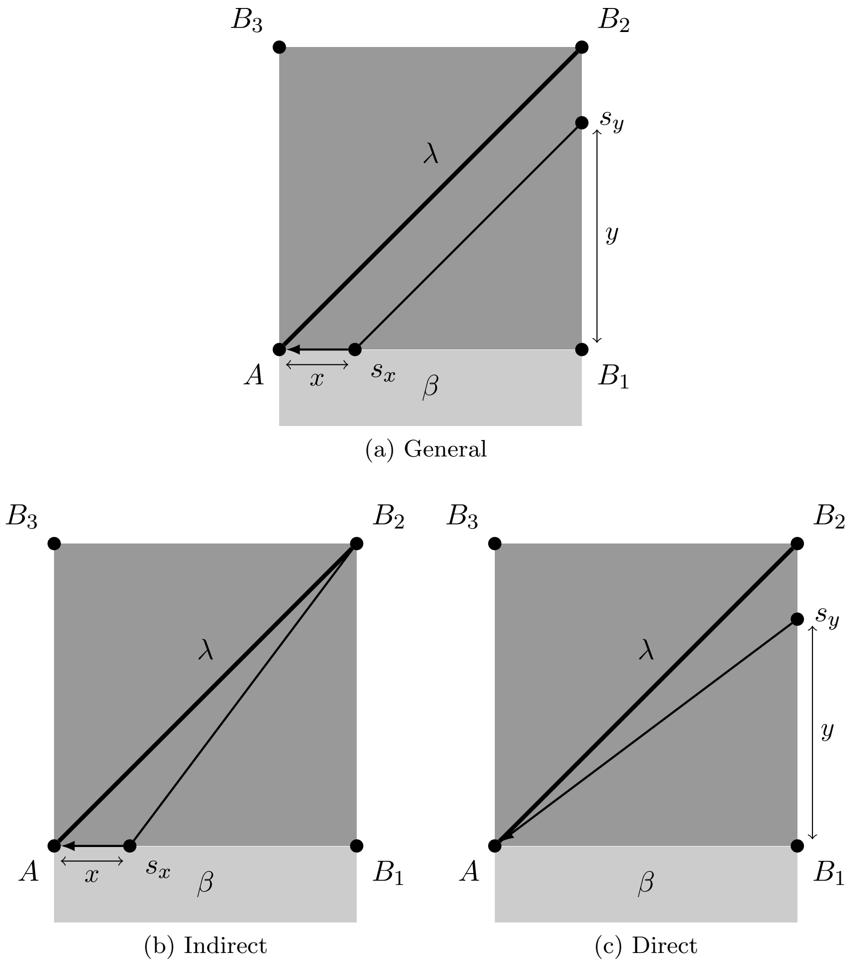

Figure 1: The (a) Genereal Field D* cost function on triangle AB_1B_2 separates into two sub-cases (b) Indirect and (c) Direct.

originating from some point s_y on edge B_1B_2, travelling through a triangle, AB_1B_2 weighted with cost λ, to a point s_x on edge AB_1 and from this point along the edge, weighted with cost β, to A. Mathematically this is expressed as:

$$g(A) = \min_{x,y \in [0,1]} [\beta x + \lambda \sqrt{(1-x)^2 + y^2} + (1-y) g(B_1) + y\, g(B_2)] \tag{1}$$

where scalar y parameterises the point s_y on edge B_1B_2 and scalar x parameterises the point s_x on edge AB_1. In particular, y parameterises a linear interpolation of the path costs of nodes B_1 and B_2, $g(B_1)$ and $g(B_2)$ respectively. Ferguson and Stentz [2] show that, when minimising Function 1 either x and y can be eliminated to produce two separate cases, the *indirect case*, shown in Figure 1b and the *direct case*, shown in Figure 1c. Only one of these cases produces an optimal shortest path across the triangle and in practice they are minimised separately, instead of Function 1. The authors also explicitly describe the two boundary conditions of the *direct case* as separate cases, but we leave these out for the sake of simplicity.

Sapronov and Lacaze [5] show that the same result holds for arbitrary non-degenerate triangles.

2 Notation

As much of the discussion in this work involves simplices, we now introduce some notation. A *simplex* generalises the concept of a triangle in two dimensions and a tetrahedron in three dimensions to arbitrary dimensions. An n-simplex is a n-dimensional *polytope* constructed from $n+1$ vertices, and is defined as the *convex hull* of those vertices.

The convex hull of any nonempty subset of the $n+1$ vertices defining the simplex is called a *face* of the simplex and is itself a simplex. An $m+1$ subset of the original $n+1$ vertices is an m-simplex and can be called an *m-face* of the n-simplex. Under this formulation, 0-faces are equivalent to *vertices*, 1-faces to *edges* and (n-1)-faces to *facets*.

The number of m-faces in an n-simplex, with $m < n$ is equal to the binomial coefficient $\binom{n+1}{m+1}$. Using this formula, it can be seen that there are $n+1$ facets in an n-simplex, for example.

Simplices may be connected together in a *Simplicial Complex*, sharing vertices and facets. In a 3D Simplicial Complex, two adjacent tetrahedra share a facet (triangle) and three vertices.

When referring to simplices, A will denote the *apex* vertex, or the node for which we are calculating the path cost $g(A)$, while the vertices $B_1 \dots B_n$ form a facet of the simplex opposite A, which we call the *base facet*. The path costs $g(B_i)\,\forall i \in (1,\dots,n)$, form a linear weighting system on the base facet. We denote the interior weight of the simplex with λ, while we use β_i to denote the weights of simplices adjacent to the simplex under consideration.

3 Separation of Field D* cases for simplices

Here we prove that for *non-degenerate simplices*, Field D* also separates into two cases, *direct* and *indirect*.

Suppose that all global minima are for a path through some P_1 in the interior of the base facet of the simplex and some $P_2, P_3, \dots, P_m$ in the interior of the side facets. An example configuration is shown in Figure 2. The path travels from $P_1 \dots P_m A$. We will show that this leads to a contradiction.

The path cost of a point P_1 is linear in P_1 by definition. We express the linear weighting for the sake of simplicity as $\mathbf{w} \cdot P_1 + d$, where $\mathbf{w}$ is a n-dimensional vector representing a linear scaling and d is the offset of this linear scaling system.

Let T be the point where AP_2 meets the face $B_2B_3 \dots B_n$. We parameterise the line segments constituting the path throught the simplex with t:

$$\begin{aligned} Q_1(t) &= T + t(P_1 - T) \\ Q_2(t) &= T + t(P_2 - T) \\ Q_i(t) &= A + u(t)(P_i - A)\,\forall i > 2 \end{aligned}$$

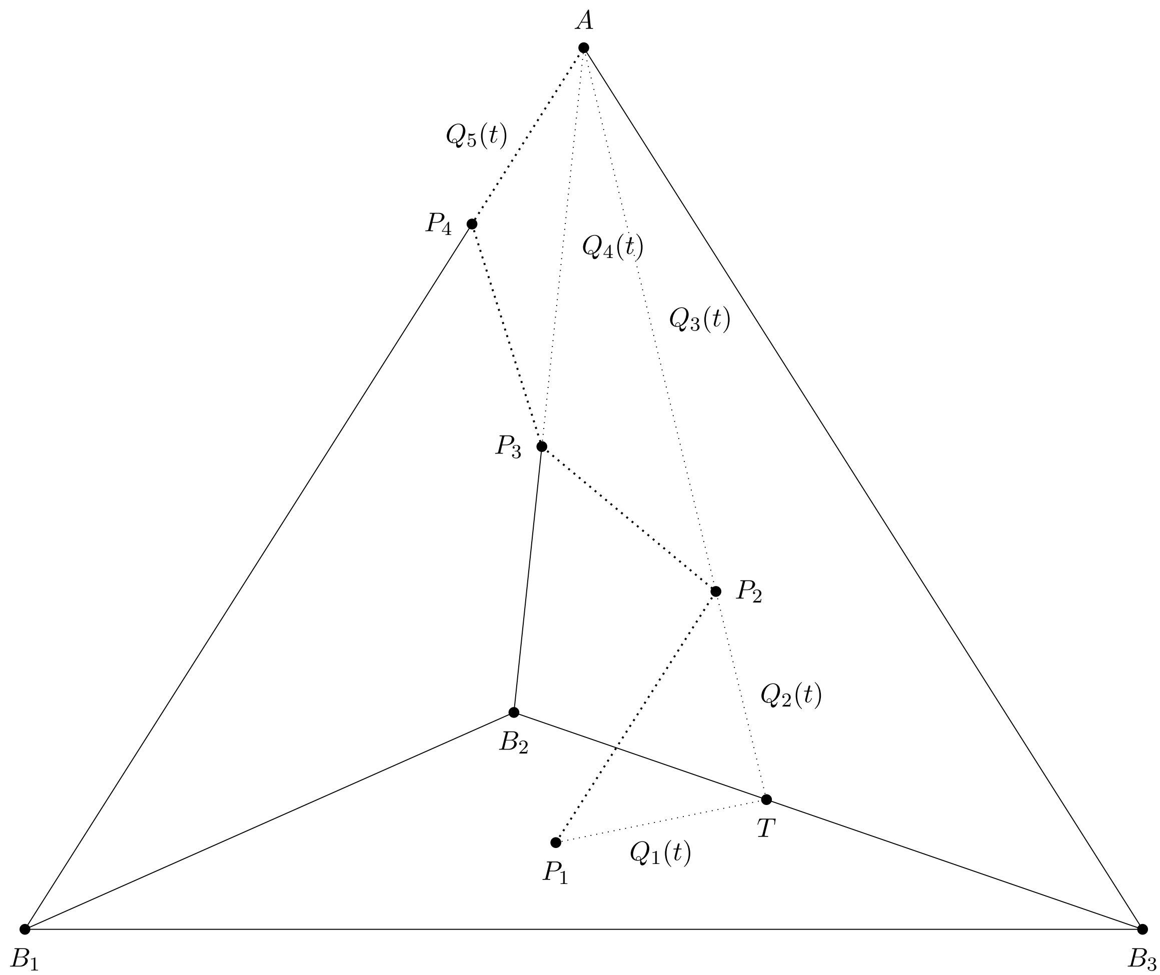

Figure 2: Depiction of the proof by contradiction, which shows that optimal paths originating on the *interior* of the base plane cannot travel on side facets, P_1T and P_2T are parameterised by t, while P_2A, P_3A and P_4A are parameterised by u. P_1, P_2, P_3 and P_4 are points on a path that is assumed to be a global minimum. Since u is linear in t, and the cost function describing a path through these points, $G(t)$ is itself linear in t, a local minimum for this function can only occur when the slope is zero. However, starting from 1, t can be adjusted downwards until P_1 and P_2 reach T on the facet boundary or P_i reaches the base facet for $i > 2$ without changing the cost, contradicting the assumption that the global minimum occurs within the base plane.

where $u(t)$ is a linear function of t that we derive from the following relation:

$$\begin{aligned} u(t)(A-P_2)+t(P_2-T) &= (A-T) \\ \Rightarrow u(t)\|A-P_2\|+t\|P_2-T\| &= \|A-T\| \\ \Rightarrow u(t) &= \frac{\|A-T\|}{\|A-P_2\|} - t\frac{\|P_2-T\|}{\|A-P_2\|} \end{aligned} \tag{2}$$

In particular, the term $\|Q_3(t)-Q_2(t)\|$ is linear in $u(t)$, and thus t, by using the relation expressed in Equation 2:

$$\begin{aligned} \|Q_3(t)-Q_2(t)\| &= \|u(t)(P_3-A)+A-t(P_2-T)-T\| \\ &= \|u(t)(P_3-A)-t(P_2-T)+(A-T)\| \\ &= \|u(t)(P_3-A)-t(P_2-T)+u(t)(A-P_2)+t(P_2-T)\| \\ &= \|u(t)(P_3-A)+u(t)(A-P_2)\| \\ &= u(t)\|P_3-P_2\| \end{aligned}$$

The other distance components of consecutive sections of the path are also linear in t:

$$\begin{aligned} \|Q_2(t)-Q_1(t)\| &= \|t(P_2-T)+T-t(P_1-T)-T\| \\ &= \|t(P_2-P_1)\| \\ &= t\|P_2-P_1\| \\ \|Q_{i+1}(t)-Q_i(t)\| &= \|u(t)(P_{i+1}-A)+A-u(t)(P_i-A)-A\| \\ &= \|u(t)(P_{i+1}-P_i)\| \\ &= u(t)\|P_{i+1}-P_i\| \\ \|A-Q_m(t)\| &= \|A-u(t)(P_m-A)-A\| \\ &= u(t)\|P_m-A\| \end{aligned}$$

Note that $P_i = Q_i(1)$. Therefore we can say that there is an open interval I including the value 1, containing a range of values for t such that $Q_i(t)$ will always lie within the interior of their respective facets. Thus, $t \in I$ will always produce a legal path.

Let $G(t)$ be the path cost through $Q_i(t)$. Since the points P_i are supposed to give the globally optimal path, $G(1)$ must be a local minimum on I. Now $G(t)$ can be expressed as:

$$\begin{aligned} G(t) =& \mathbf{w}\cdot Q_1(t)+d+\lambda\|Q_2(t)-Q_1(t)\|+\beta_1\|Q_3(u(t))-Q_2(u(t))\|+ \\ & \ldots+\beta_o\|Q_m(u(t))-A\| \\ =& \mathbf{w}\cdot Q_1(t)+d+\lambda t\|P_2-P_1\|+\beta_1 u(t)\|P_3-P_2\|+ \\ & \ldots+\beta_n u(t)\|P_m-A\| \end{aligned}$$

Thus, G is a linear function of t. A linear function can only have a local minimum on an open interval if its slope is zero. But in that case, we can start with $t=1$ and then adjust it upwards or downwards until any $Q_i(t)$ reaches the boundary of its corresponding facet without changing the cost because of the zero slope. But this contradicts the assumption that there are no global minima except where P_i are in the interior.

This proves that it is not possible for the path with the lowest cost to include points on both the interior of the base facet and the side facets. Therefore, in the direct case, the path must travel from a point on the base facet directly to the apex node A. However, it is possible for the shortest path to originate from points on the *boundary* [1] of the base facet and then travel to points on the side facets. These form the indirect cases in higher dimensions. In particular, we have not proved that the indirect cases may not have more than two path segments.

References

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[3] P.E. Hart, N.J. Nilsson, and B. Raphael. A Formal Basis for the Heuristic Determination of Minimum Cost Paths. *IEEE Transactions on Systems Science and Cybernetics*, 4(2):100–107, 1968.

[4] Nils J. Nilsson. *Principles of Artificial Intelligence*. Morgan Kaufmann Publishers, January 1993.

[5] Lenny Sapronov and Alberto Lacaze. Path planning for robotic vehicles using Generalized Field D*. volume 6962, pages 69621C – 69621C–12. SPIE, 2008.

[1] In 3D, boundary of the triangle that forms the base facet would consist of the triangle edges. In 4D, the boundary of the tetrahedron that forms the base facet, would itself consist of triangles.