

## What Does Entailment for PTL Mean?

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### Abstract

We continue recent investigations into the problem of reasoning about typicality. We do so in the framework of Propositional Typicality Logic (PTL), which is obtained by enriching classical propositional logic with a typicality operator and characterized by a preferential semantics *à la* KLM. In this paper we study different notions of entailment for PTL. We take as a starting point the notion of Rational Closure defined for KLM-style conditionals. We show that the additional expressivity of PTL results in different versions of Rational Closure for PTL — versions that are equivalent with respect to the conditional language originally proposed by KLM.

desiderata that any notion of entailment deemed as appropriate with regard to typicality ought to satisfy. Central to these desiderata is the notion of Rational Closure as defined by Lehmann and Magidor for KLM-style conditionals in a propositional setting (Lehmann and Magidor 1992). We show that different definitions of Rational Closure, that are equivalent with respect to the conditional language originally proposed by Kraus et al., in the framework of PTL result in distinct kinds of closure. Finally we prove that it is not possible to define in PTL a notion of entailment that satisfies all our desiderata.

## 1 Introduction

Propositional Typicality Logic (PTL) (Booth, Meyer, and Varzinczak 2012; 2013) is a recently proposed logic allowing for the representation of and reasoning with a notion of *typicality*. It is obtained by enriching classical propositional logic with a *typicality operator*  $\bullet$ , the intuition of which is to capture the most typical (or normal) situations (worlds) in which a given formula holds.

PTL is characterised using a preferential semantics similar to that proposed by Shoham (1988) and extensively developed by Kraus et al. (1990) and by Lehmann and Magidor (1992; 1995). It can also be viewed as an enriched version of the conditional logic originally studied by Kraus and colleagues.

In spite of the nonmonotonic features introduced by the adoption of a preferential semantics, the obvious definition of entailment for PTL (i.e., a Tarski-style consequence operator) remains monotonic. Such a notion of entailment is inappropriate in nonmonotonic contexts, in particular when reasoning about typicality.

In this paper we investigate different, alternative notions of entailment for PTL. We start by proposing a set of

## 2 Background

### Logical preliminaries

We work in a propositional language over a finite set of propositional *atoms*  $\mathcal{P}$ . In later sections we adopt a richer language. We use  $p, q, \dots$  as meta-variables for atoms. Propositional formulae (and in later sections, formulae of the richer language) are denoted by  $\alpha, \beta, \dots$ , and are recursively defined in the usual way:  $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha$ . All the other Boolean truth-functional connectives ( $\vee, \rightarrow, \leftrightarrow, \dots$ ) are defined in terms of  $\neg$  and  $\wedge$  in the standard way. We use  $\top$  as an abbreviation for  $p \vee \neg p$ , and  $\perp$  for  $p \wedge \neg p$ , for some  $p \in \mathcal{P}$ . With  $\mathcal{L}$  we denote the set of all propositional formulae.

We denote by  $\mathcal{U}$  the set of all *valuations*  $v : \mathcal{P} \longrightarrow \{0, 1\}$ . Sometimes we shall represent the valuations of the logic under consideration as sets of literals (i.e., atoms or negated atoms), which each literal indicating the truth-value of the respective atom. Thus, for the logic generated from  $\mathcal{P} = \{p, q\}$ , the valuation in which  $p$  is true and  $q$  is false will be represented as  $\{p, \neg q\}$ . Satisfaction of a formula  $\alpha \in \mathcal{L}$  by  $v \in \mathcal{U}$  is defined in the usual truth-functional way and is denoted by  $v \models \alpha$ .

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## KLM-style conditional logic

As indicated above, our proposal is an expressivity enrichment of the preferential conditional logic proposed by (Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992; Lehmann 1995), often referred to as the KLM approach. The conditional language used in such an approach is based on conditional expressions  $\alpha \sim \beta$ , read as ‘Typically, if  $\alpha$  then  $\beta$ ’. Reasoning in such a conditional framework means to be able, given a set of conditionals, to derive new conditionals. KLM (Kraus, Lehmann, and Magidor 1990) provide the following list of formal properties that the conditional  $\sim$  ought to satisfy:

$$\begin{array}{ll}
 \text{(Ref)} \quad \alpha \sim \alpha & \text{(LLE)} \quad \frac{\models \alpha \leftrightarrow \beta, \alpha \sim \gamma}{\beta \sim \gamma} \\
 \text{(And)} \quad \frac{\alpha \sim \beta, \alpha \sim \gamma}{\alpha \sim \beta \wedge \gamma} & \text{(Or)} \quad \frac{\alpha \sim \gamma, \beta \sim \gamma}{\alpha \vee \beta \sim \gamma} \\
 \text{(RW)} \quad \frac{\alpha \sim \beta, \models \beta \rightarrow \gamma}{\alpha \sim \gamma} & \text{(CM)} \quad \frac{\alpha \sim \beta, \alpha \sim \gamma}{\alpha \wedge \beta \sim \gamma}
 \end{array}$$

A conditional satisfying such properties is called a *preferential conditional*. We can add another desired property:

$$\text{(RM)} \quad \frac{\alpha \sim \gamma, \alpha \not\sim \neg \beta}{\alpha \wedge \beta \sim \gamma}$$

A preferential conditional that satisfies also (RM) is called a *rational conditional*. For a better understanding of the reasons behind such desiderata and of the correlated semantic constructions, the reader is referred to the work by Kraus et al. (1990) and Lehmann and Magidor (1992).

## Propositional Typicality Logic

The language of PTL, denoted by  $\mathcal{L}^\bullet$ , is recursively defined as follows:

$$\alpha ::= p \mid \neg \alpha \mid \alpha \wedge \alpha \mid \bullet \alpha$$

where, as before,  $p$  denotes an atom and all the other connectives are defined in terms of  $\neg$  and  $\wedge$ , and  $\top$  and  $\perp$  are seen as abbreviations.

Intuitively, a sentence of the form  $\bullet \alpha$  is understood to refer to the typical situations in which  $\alpha$  holds. (Note that  $\alpha$  can itself be a  $\bullet$ -sentence.)

Let  $\mathcal{P} = \{b, f, p, o\}$ , where the propositions  $b, f, p$  and  $o$  stand for, respectively, ‘‘Tweety is a bird’’, ‘‘Tweety flies’’, ‘‘Tweety is a penguin’’ and ‘‘Tweety is an ostrich’’. The following are examples of  $\mathcal{L}^\bullet$ -sentences:  $\bullet b, o \rightarrow \neg \bullet b, (p \vee o) \leftrightarrow (b \wedge \bullet \neg f)$ .

The semantics of PTL is in terms of structures we call *ranked interpretations*.

**Definition 1 (Ranked Interpretation)** *A ranked interpretation  $\mathcal{R}$  is a pair  $\langle \mathcal{V}, < \rangle$ , where  $\mathcal{V} \subseteq \mathcal{U}$  and  $< \subseteq \mathcal{V} \times \mathcal{V}$  is a modular order over  $\mathcal{V}$ .*

Observe that modular orders can be obtained from total preorders by imposing anti-symmetry.

Given a set  $X$ ,  $< \subseteq X \times X$  is modular if and only if there is a ranking function  $rk : X \rightarrow \mathbb{N}$  s.t. for every  $x, y \in X$ ,  $x < y$  iff  $rk(x) < rk(y)$ . Given a ranked interpretation  $\mathcal{R}$ ,

the intuition is that valuations lower down in the ordering are more preferred (or deemed more normal or typical) than those higher up. We can write a ranked interpretation  $\mathcal{R} = \langle \mathcal{V}, < \rangle$  alternatively as a partition  $\mathcal{R} = (L_0, \dots, L_n)$  of  $\mathcal{V}$ , where  $v < v'$  iff  $v \in L_i, v' \in L_j$  and  $i < j$ . That is,  $L_i$  is the set of all valuations of rank  $i$ . Given a ranked interpretation  $\mathcal{R} = \langle \mathcal{V}, < \rangle$  and a formula  $\alpha \in \mathcal{L}^\bullet$ , every valuation  $v$  in  $\mathcal{V}$  either satisfies or does not satisfy  $\alpha$ . The notion of satisfaction is defined inductively in the classical way, adding the following condition

$$v \Vdash \bullet \alpha \text{ iff } v \Vdash \alpha \text{ and there is no } v' < v \text{ s.t. } v' \Vdash \alpha$$

With  $\llbracket \alpha \rrbracket^{\mathcal{R}}$  we denote the set of all valuations satisfying  $\alpha$  in a ranked interpretation  $\mathcal{R}$ . Given  $\alpha \in \mathcal{L}^\bullet$  and  $\mathcal{R}$  a ranked interpretation, we say that  $\alpha$  is *satisfiable* in  $\mathcal{R}$  if  $\llbracket \alpha \rrbracket^{\mathcal{R}} \neq \emptyset$ , otherwise  $\alpha$  is *unsatisfiable* in  $\mathcal{R}$ . We say that  $\alpha$  is *true* in  $\mathcal{R}$  (denoted as  $\mathcal{R} \Vdash \alpha$ ) if  $\llbracket \alpha \rrbracket^{\mathcal{R}} = \mathcal{V}$ . A *knowledge base*  $\mathcal{K}$  is a finite set of formulae  $\mathcal{K} \subseteq \mathcal{L}^\bullet$ , and  $\mathcal{R} \Vdash \mathcal{K}$  if  $\mathcal{R} \Vdash \alpha$  for every  $\alpha \in \mathcal{K}$ .

Figure 1 depicts an example of ranked interpretation for  $\mathcal{P} = \{b, f, p\}$ .

$\mathcal{R}$	$L_2 :$	$\{b, f, p\}$
	$L_1 :$	$\{b, \neg f, \neg p\} \quad \{b, \neg f, p\}$
	$L_0 :$	$\{\neg b, \neg f, \neg p\} \quad \{\neg b, f, \neg p\} \quad \{b, f, \neg p\}$

Figure 1: An  $\mathcal{L}^\bullet$ -interpretation for  $\mathcal{P} = \{b, f, p\}$ .

In the ranked interpretation  $\mathcal{R}$  of Figure 1, we have  $\llbracket \bullet b \rrbracket^{\mathcal{R}} = \{\{b, f, \neg p\}\}$ ,  $\llbracket \bullet p \rrbracket^{\mathcal{R}} = \{\{b, \neg f, p\}\}$  and  $\llbracket \bullet (b \wedge \neg f) \rrbracket^{\mathcal{R}} = \{\{b, \neg f, \neg p\}, \{b, \neg f, p\}\}$ .

A useful property of the typicality operator  $\bullet$  is that it allows us to express KLM-style conditionals. Given a ranked interpretation  $\mathcal{R} = \langle \mathcal{V}, < \rangle$ , for  $\alpha, \beta \in \mathcal{L}$ ,  $\mathcal{R} \Vdash \alpha \sim \beta$  iff all the  $<$ -minimal valuations satisfying  $\alpha$  also satisfy  $\beta$ .

**Proposition 1 (Booth et al. (2013))** *Let  $\mathcal{R} = \langle \mathcal{V}, < \rangle$ . For every  $\alpha, \beta \in \mathcal{L}$  (i.e.,  $\alpha$  and  $\beta$  are propositional formulae),  $\mathcal{R} \Vdash \alpha \sim \beta$  if and only if  $\mathcal{R} \Vdash \bullet \alpha \rightarrow \beta$ .*

In other words, the KLM framework can be embedded in PTL. The converse does not hold, as witnessed by the following result.

**Proposition 2** *There are  $\mathcal{L}^\bullet$ -sentences that cannot be expressed as a set of KLM-style  $\sim$ -statements.*

Hence, PTL does indeed add to the expressivity of the KLM approach. However, nesting of the typicality operator does not increase the expressivity any further, provided that we are allowed to add new propositional atoms (Booth, Meyer, and Varzinczak 2013, Theorem 18) via renaming.

**Definition 2** *We say a formula  $\alpha \in \mathcal{L}^\bullet$  is in normal form iff it is of the form  $(\bullet \theta_1 \wedge \dots \wedge \bullet \theta_t) \rightarrow (\phi \wedge (\bullet \psi_1 \vee \dots \vee \bullet \psi_s))$ , where  $t, s \geq 0$  and the  $\theta_k, \phi$  and  $\psi_k$  are all purely propositional formulae.*

It can be shown that every formula in  $\mathcal{L}^\bullet$  is equivalent to one in normal form, i.e., the normal form is complete for  $\mathcal{L}^\bullet$ . This fact will be exploited in Section 5.

We have seen in Proposition 1 that rational conditionals for *propositional* logic can be expressed in PTL. The representation result below, that extends, for  $\mathcal{L}^\bullet$ , Theorem 3.12 of Lehmann and Magidor (1992), shows that the formalisation of the KLM-style rational conditional  $\sim$  inside PTL is appropriate.

**Theorem 1** *Let  $\mathcal{R}$  be a ranked interpretation and let  $\sim_{\mathcal{R}} := \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L}^\bullet \text{ and } \alpha \sim_{\mathcal{R}} \beta\}$ . Then  $\sim_{\mathcal{R}}$  is a rational conditional. Conversely, for every rational conditional  $\sim$  there exists a ranked interpretation  $\mathcal{R}$  such that  $\sim_{\mathcal{R}} = \sim$ .*

For more details on PTL and the aforementioned properties, the reader is referred to the book chapter by Booth et al. (2013).

### 3 The entailment problem for PTL

This section provides a more formal motivation for the rest of the paper. From the perspective of knowledge representation and reasoning (KR&R), a central issue is that of what it means for a PTL sentence to *follow* from a (finite) knowledge base  $\mathcal{K}$ . An obvious approach to the matter is to follow the approach to entailment advocated by Tarski (1956) and largely adopted in the logic-based KR&R community. Let  $\text{Mod}(\mathcal{K})$  indicate the set of ranked interpretations satisfying  $\mathcal{K}$ , which we refer to as the *ranked models* of  $\mathcal{K}$ .

**Definition 3 (Ranked Entailment)** *Let  $\mathcal{K} \subseteq \mathcal{L}^\bullet$  and  $\alpha \in \mathcal{L}^\bullet$ . Then  $\mathcal{K}$  ranked-entails  $\alpha$  (noted  $\mathcal{K} \models_0 \alpha$ ) if and only if for every ranked interpretation  $\mathcal{R} \in \text{Mod}(\mathcal{K})$ ,  $\mathcal{R} \models \alpha$ .*

Given the notion of ranked entailment above, its associated consequence operator is defined as expected in the following way.

**Definition 4 (Ranked Consequence)** *Let  $\mathcal{K} \subseteq \mathcal{L}^\bullet$ .  $\text{Cn}_0(\mathcal{K}) \equiv_{\text{def}} \{\alpha \in \mathcal{L}^\bullet \mid \mathcal{K} \models_0 \alpha\}$ .*

To see why this notion of consequence is not appropriate in the context of PTL, consider the following definition of a conditional induced from a PTL knowledge base.

**Definition 5 (Induced  $\sim$ )** *Let  $\mathcal{X} \subseteq \mathcal{L}^\bullet$  be an arbitrary (possibly infinite) set of formulae. Then  $\sim_{\mathcal{X}} := \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L} \text{ and } \bullet\alpha \rightarrow \beta \in \mathcal{X}\}$ .*

It is reasonable to expect the conditional  $\sim_{\text{Cn}_0(\mathcal{K})}$  to be rational, i.e., to satisfy all the KLM properties for rationality. However, the following proposition, which mimics a similar result by KLM in the propositional case, shows that this is not the case:

**Proposition 3 (Booth et al. (2013))**  *$\sim_{\text{Cn}_0(\mathcal{K})}$  is a preferential conditional, but is not necessarily a rational conditional.*

Hence, ranked consequence as defined above, delivers an induced defeasible consequence that is preferential but that needs not be rational. This witnesses against ranked entailment as an appropriate notion of entailment for PTL.

A concrete argument against ranked consequence is given by the following example.

**Example 1** *First of all, we would like to have an entailment relation that satisfies the presumption of typicality. Informally, this means we assume for every situation that it is as*

*typical as possible. Let  $\mathcal{K}_1 := \{\mathbf{p} \rightarrow \mathbf{b}, \bullet\mathbf{b} \rightarrow \mathbf{f}\}$  (“penguins are birds” and “birds typically fly”). Given just this information about birds and penguins, it is reasonable to assume that penguins are typical birds and expect  $\bullet\mathbf{p} \rightarrow \mathbf{f}$  to follow from  $\mathcal{K}_1$  (Sections 4 and 6 contain a formalization of this principle.). It is not hard to see that with ranked consequence this requirement is not met. As a second desideratum, and strictly related to the first one, we would like our entailment relation also to be defeasible, that is, the conclusions derived under the presumption of typicality can be retracted in case of new conflicting information. For example, even if  $\bullet\mathbf{p} \rightarrow \mathbf{f}$  could follow from  $\mathcal{K}_1$ , let  $\mathcal{K}_2 := \mathcal{K}_1 \cup \{\bullet\mathbf{p} \rightarrow \neg\mathbf{f}\}$  (add to  $\mathcal{K}_1$  that “penguins typically do not fly”), we are now in a situation in which we are informed that penguins are not typical birds (they do not fly), and now we do not want  $\bullet\mathbf{p} \rightarrow \mathbf{f}$  to follow from  $\mathcal{K}_2$ . With an entailment relation based on ranked entailment, also the requirement of defeasibility is violated.*

### Rational closure of conditional KBs

In the restricted setting of conditional knowledge bases, Lehmann and Magidor (1992) proposed the *rational closure* construction. Their idea was to define a preference relation  $\preceq_{\text{LM}}$  over the set of possible ranked interpretations and then to base entailment on choosing only the most preferred, i.e., minimal, ranked models of the KB  $\mathcal{K}$ . The relation  $\preceq_{\text{LM}}$  can be perhaps most easily described using the representations of ranked interpretations in terms of sequences  $(L_1, \dots, L_n)$ . For any pair of ranked interpretations  $\mathcal{R}_1 = (L_1, \dots, L_n)$  and  $\mathcal{R}_2 = (M_1, \dots, M_n)$  (we can assume they are of the same length, fill up the tail with  $\emptyset$  otherwise), we set

$$\mathcal{R}_1 \preceq_{\text{LM}} \mathcal{R}_2 \text{ iff } \begin{array}{l} \text{either } L_i = M_i \text{ for all } i \\ \text{or for the first } j \text{ s.t. } L_j \neq M_j \\ \text{we have } L_j \supseteq M_j \end{array}$$

This is not exactly the way it was defined by Lehmann and Magidor, but this representation can easily be derived from other work on rational closure such as that of Booth and Paris (1998) and Giordano et al. (2012). The idea is that those ranked interpretations should be preferred in which as many valuations as possible are judged to be as plausible as  $\mathcal{K}$  allows.

Clearly  $\preceq_{\text{LM}}$  forms a partial order over ranked interpretations. Lehmann and Magidor showed that, for conditional  $\mathcal{K}$ , there exists a unique  $\preceq_{\text{LM}}$ -minimum element  $\mathcal{R}^{\text{rc}}(\mathcal{K})$  among all the ranked interpretations of  $\mathcal{K}$ . We will refer to this element as the *LM-minimum*. Then the rational closure of  $\mathcal{K}$  is the relation  $\sim_{\mathcal{R}^{\text{rc}}(\mathcal{K})}$ . In Section 5 we will see what happens when we try to extend this construction to arbitrary knowledge bases in  $\mathcal{L}^\bullet$ .

### 4 Towards a notion of entailment for PTL

We have seen that ranked entailment has some drawbacks. Therefore, the question as to what logical consequence in PTL should mean remains mostly unanswered. In this section we first specify and discuss a list of properties that, at first glance, seem reasonable for an appropriate notion of entailment in the context of PTL. In the subsequent section we

consider specific alternatives to ranked entailment and check them against our desiderata.

We start by introducing some notation. With  $\models_{\mathcal{R}} \subseteq \mathcal{P}(\mathcal{L}^\bullet) \times \mathcal{L}^\bullet$ , we denote any entailment relation on the language of PTL. Given an entailment relation  $\models_{\mathcal{R}}$ , its associated *consequence* operator is defined in the usual way as follows, where  $\mathcal{K} \subseteq \mathcal{L}^\bullet$ :

$$Cn_{\mathcal{R}}(\mathcal{K}) \equiv_{\text{def}} \{\alpha \in \mathcal{L}^\bullet \mid \mathcal{K} \models_{\mathcal{R}} \alpha\}$$

The obvious starting point is to consider some properties of consequence operators as proposed by Tarski (1956). The first two of them are:

**P1**  $\mathcal{K} \subseteq Cn_{\mathcal{R}}(\mathcal{K})$  (Inclusion)

**P2**  $Cn_{\mathcal{R}}(\mathcal{K}) = Cn_{\mathcal{R}}(Cn_{\mathcal{R}}(\mathcal{K}))$  (Idempotency)

A justification for Inclusion is quite obvious: whatever is in a knowledge base, it must be amongst its consequences. Idempotency, also known as Iteration, specifies that a consequence operator behaves as a ‘once-off’ operation, in the same spirit as that of a closure operator. There is an agreement in the literature that both Inclusion and Idempotency are desirable properties to have.

Ranked entailment, as defined in Section 3, satisfies Properties P1–P2. However,  $Cn_0(\cdot)$  also satisfies the classical property of Monotonicity: If  $\mathcal{K} \subseteq \mathcal{K}'$ , then  $Cn_0(\mathcal{K}) \subseteq Cn_0(\mathcal{K}')$ . As seen in Example 1, this is a property that we do not want  $Cn_{\mathcal{R}}(\cdot)$  to satisfy (at least not in general).

So, we require  $Cn_{\mathcal{R}}(\cdot)$  to be a *nonmonotonic* consequence operator. Traditionally, this amounts to requiring  $Cn_{\mathcal{R}}(\cdot)$  to satisfy the following two properties:

**P3**  $Cn_0(\mathcal{K}) \subseteq Cn_{\mathcal{R}}(\mathcal{K})$  (Ampliativeness)

**P4** For some  $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{L}^\bullet$ ,  $\mathcal{K} \subseteq \mathcal{K}'$  but  $Cn_{\mathcal{R}}(\mathcal{K}) \not\subseteq Cn_{\mathcal{R}}(\mathcal{K}')$  (Defeasibility)

Ampliativeness, sometimes referred to as supra-classicality, namely when the basic underlying entailment relation is classical, says that  $Cn_{\mathcal{R}}(\cdot)$  should be more ‘venturous’ than its underlying ranked entailment. In Example 1, we have  $\bullet p \rightarrow f \notin Cn_0(\mathcal{K}_1)$ , i.e., it does not follow that “penguins typically fly”. However, given the information in  $\mathcal{K}_1$ , a case can be made for having  $\bullet p \rightarrow f$  as a plausible (though provisional) conclusion, e.g. in the absence of information to the contrary.

Defeasibility specifies that  $Cn_{\mathcal{R}}(\cdot)$  should be flexible enough to disallow previously derived conclusions in the light of new (possibly conflicting) information. In Example 1, assuming  $\bullet p \rightarrow f \in Cn_{\mathcal{R}}(\mathcal{K}_1)$  is the case, then  $\bullet p \rightarrow f$  should no longer be concluded if  $\bullet p \rightarrow \neg f$  is added to  $\mathcal{K}_1$ .

Similarly to Lehmann and Magidor in the propositional case, we would ideally like the defeasible inference relation associated with  $Cn_{\mathcal{R}}(\mathcal{K})$  to satisfy all the KLM rationality properties:

**P5**  $\sim_{Cn_{\mathcal{R}}(\mathcal{K})}$  is a rational conditional on  $\mathcal{L}$  (Conditional Rationality)

The following “single model” property can be straightforwardly shown to be a strengthening of P5:

**P6** For every  $\mathcal{K} \subseteq \mathcal{L}^\bullet$ , there is a ranked interpretation  $\mathcal{R}$  such that  $\mathcal{R} \models \mathcal{K}$  and, for all  $\alpha \in \mathcal{L}^\bullet$ ,  $\alpha \in Cn_{\mathcal{R}}(\mathcal{K})$  if and only if  $\mathcal{R} \models \alpha$  (Single Model)

In the special case when  $\mathcal{K}$  is a (propositional) conditional knowledge base (i.e., when  $\mathcal{K}$  is of the form  $\{\bullet \alpha \rightarrow \beta \mid \alpha, \beta \in \mathcal{L}\}$ ), the result should coincide with Lehmann and Magidor’s definition of rational closure:

**P7** If  $\mathcal{K}$  is a conditional knowledge base, then  $\sim_{Cn_{\mathcal{R}}(\mathcal{K})} = \sim_{\mathcal{R}^{\text{rc}}(\mathcal{K})}$  (Extends Rational Closure)

Finally, the following two properties were shown by Lehmann and Magidor to be satisfied by the rational closure for conditional knowledge bases. They say that  $Cn_{\mathcal{R}}(\cdot)$  should coincide with ranked entailment for certain restricted classes of sentences.

**P8** Let  $\alpha \in \mathcal{L}$ . Then  $\alpha \in Cn_{\mathcal{R}}(\mathcal{K})$  if and only if  $\alpha \in Cn_0(\mathcal{K})$  (Strict Entailment)

**P9** Let  $\alpha \in \mathcal{L}$ . Then  $\bullet \top \rightarrow \alpha \in Cn_{\mathcal{R}}(\mathcal{K})$  if and only if  $\bullet \top \rightarrow \alpha \in Cn_0(\mathcal{K})$  (Typical Entailment)

## 5 LM-entailment

We now come to our first construction of a consequence operator in PTL. The idea is to try to lift the rational closure construction from conditional knowledge bases to arbitrary knowledge bases in  $\mathcal{L}^\bullet$ . As a promising sign, we first observe that there is nothing to stop us from using the preference relation  $\leq_{\text{LM}}$  to compare ranked interpretations of *any* PTL knowledge base  $\mathcal{K}$ . The question is, does there always exist a *unique* LM-minimum element of the ranked models of  $\mathcal{K}$ , as there does in the restricted conditional case? And if so, how can we construct it? We answer these questions in this section.

We assume as input a PTL knowledge base  $\mathcal{K} = \{\alpha_1, \dots, \alpha_n\}$ . We assume that each sentence  $\alpha_j$  is in normal form (see Definition 2). For any ranked interpretation  $\mathcal{R} = \langle \mathcal{V}, < \rangle$  and  $S \subseteq \mathcal{V}$ , we define  $\mathcal{R} \downarrow S$  (the restriction of  $\mathcal{R}$  to  $S$ ) as  $\langle \mathcal{V} \cap S, < \cap (S \times S) \rangle$ .

We construct a sequence  $(\mathcal{R}_0, \mathcal{R}_1, \dots)$  of ranked interpretations as follows, where  $\mathcal{R}_i = \langle \mathcal{U}, <_i \rangle$  (i.e., the set of valuations  $\mathcal{V}$  is always the full set of all valuations):

1. Initialise  $<_0 := \emptyset$   
*start with an initial ranked interpretation in which all valuations are equally preferred.*
2.  $S_{i+1} := \llbracket \mathcal{K} \rrbracket^{\mathcal{R}_i}$   
*separate the valuations which satisfy  $\mathcal{K}$  w.r.t. the current ranked interpretation  $\mathcal{R}_i$  from those that do not.*
3. If  $S_{i+1} = S_i$  then STOP and return  $\mathcal{R}^*(\mathcal{K}) = \mathcal{R}_i \downarrow S_{i+1}$ .  
*if the division is the same as in the previous round then eliminate completely from the current ranked interpretation those valuations that do not satisfy  $\mathcal{K}$  w.r.t.  $\mathcal{R}_i$  and return the interpretation that remains.*
4. Otherwise  $<_{i+1} := <_i \cup (S_{i+1} \times S_{i+1}^c)$ ,  $i := i + 1$  and go to 2.  
*otherwise create a new ranked interpretation  $\mathcal{R}_{i+1}$  by making every valuation not in  $S_{i+1}$  less plausible than every valuation in  $S_{i+1}$ . (Note  $S^c$  denotes  $\mathcal{U} \setminus S$ .)*

**Example 2** Assume  $\mathcal{P} = \{p, q, t\}$  and that the knowledge base  $\mathcal{K}$  is composed by the formulae:

$$\bullet\top \rightarrow (\neg p \wedge \neg q), \quad \bullet p \rightarrow \bullet t, \quad \bullet q \rightarrow \bullet \neg t$$

The procedure initialises with  $<_0 = \emptyset$ . The only valuations that satisfy all three formulae w.r.t.  $\mathcal{R}_0$  are those satisfying both  $\neg p$  and  $\neg q$ . Thus  $S_1 = \llbracket \mathcal{K} \rrbracket^{\mathcal{R}_0} = \{\{\neg p, \neg q, t\}, \{\neg p, \neg q, \neg t\}\}$  and so we obtain  $\mathcal{R}_1$  by setting  $<_1$  to be the 2-layer modular order with  $S_1$  as the lower layer. Note that  $\llbracket \bullet t \rrbracket^{\mathcal{R}_1} = \{\{\neg p, \neg q, t\}\}$  and  $\llbracket \bullet \neg t \rrbracket^{\mathcal{R}_1} = \{\{\neg p, \neg q, \neg t\}\}$ , so we can see that none of valuations in  $S_1^c$  are able to satisfy either  $\bullet p \rightarrow \bullet t$  or  $\bullet q \rightarrow \bullet \neg t$  w.r.t.  $\mathcal{R}_1$ . As a consequence  $S_2 = \llbracket \mathcal{K} \rrbracket^{\mathcal{R}_1} = S_1$  and so the procedure terminates here with  $\mathcal{R}^*(\mathcal{K}) = \mathcal{R}_1 \downarrow S_2$ . That is,  $\mathcal{R}^*(\mathcal{K})$  is the ranked interpretation consisting of just a single layer containing  $\{\neg p, \neg q, t\}$  and  $\{\neg p, \neg q, \neg t\}$ .

We now need to show a number of things: (i) The algorithm will always terminate, (ii) it returns a ranked interpretation which furthermore satisfies  $\mathcal{K}$ , and (iii) for any other ranked model of  $\mathcal{K}$  we have  $\mathcal{R}^*(\mathcal{K}) \leq_{\text{LM}} \mathcal{R}$ .

We know the following about the first two items:

**Lemma 1** Assuming each sentence in  $\mathcal{K}$  is in normal form, the following hold for each  $i \geq 0$ :

- (i).  $S_i \subseteq S_{i+1}$ , i.e.,  $\llbracket \mathcal{K} \rrbracket^{\mathcal{R}_{i-1}} \subseteq \llbracket \mathcal{K} \rrbracket^{\mathcal{R}_i}$ .
- (ii). For all  $v_1, v_2 \in \mathcal{U}$ , if  $v_1 <_i v_2$  then  $v_1 \in \llbracket \mathcal{K} \rrbracket^{\mathcal{R}_i}$ .
- (iii).  $\mathcal{R}_{i+1}$  is a ranked interpretation, i.e.,  $<_{i+1}$  is a modular order.

From part (i) above we know that the algorithm terminates, since it generates a sequence of ranked (by part (iii)) interpretations in which the set of valuations satisfying  $\mathcal{K}$  increases monotonically from one ranked interpretation to the next. Since each ranked interpretation is finite the stopping criterion in step 3 of the algorithm is guaranteed to occur eventually.

To show that the algorithm returns a ranked model of  $\mathcal{K}$  it suffices to show the following.

**Lemma 2** Assuming each sentence in  $\mathcal{K}$  is in normal form, for each  $i \geq 0$ ,  $\mathcal{R}_i \downarrow S_{i+1}$  is a model of  $\mathcal{K}$ .

So at each stage of the algorithm, the current ranked interpretation, when those valuations not satisfying  $\mathcal{K}$  are excluded, forms a ranked interpretation of  $\mathcal{K}$ . Since the output  $\mathcal{R}^*(\mathcal{K})$  of the algorithm takes precisely this form we have the following result.

**Theorem 2** Assuming each sentence in  $\mathcal{K}$  is in normal form, we have  $\mathcal{R}^*(\mathcal{K}) \Vdash \mathcal{K}$ .

Next we want to show that for any other ranked interpretation of  $\mathcal{K}$  we have  $\mathcal{R}^*(\mathcal{K}) \leq_{\text{LM}} \mathcal{R}$ . Let  $\mathcal{R}^*(\mathcal{K}) = (S_1, \dots, S_m)$  and let  $\mathcal{R} = (T_1, \dots, T_m)$  be any other ranked interpretation of  $\mathcal{K}$ . If one of the two sequences is shorter than the other, we simply fill its tail with an appropriate number of empty sets to ensure the sequences have equal length.

**Lemma 3** Let  $i \geq 1$ . If  $T_j = S_j$  for all  $j < i$  then  $T_i \subseteq S_i$ .

From this lemma we can state:

**Theorem 3** Assume each sentence in  $\mathcal{K}$  is in normal form and let  $\mathcal{R}$  be a ranked interpretation such that  $\mathcal{R} \Vdash \mathcal{K}$ . Then  $\mathcal{R}^*(\mathcal{K}) \leq_{\text{LM}} \mathcal{R}$ .

We will denote by  $Cn_{\text{LM}}(\cdot)$  the consequence operator defined via  $\mathcal{R}^*(\mathcal{K})$ , i.e.,  $Cn_{\text{LM}}(\mathcal{K}) = \{\alpha \in \mathcal{L}^\bullet \mid \mathcal{R}^*(\mathcal{K}) \Vdash \alpha\}$ . The next result outlines which properties from the previous section are satisfied by  $Cn_{\text{LM}}(\cdot)$ .

**Proposition 4**  $Cn_{\text{LM}}(\cdot)$  satisfies Inclusion, Ampliativeness, Defeasibility, Conditional Rationality, Single Model, Extends Rational Closure and Typical Entailment, but **not** Strict Entailment.

## 6 Different notions of minimality

As we have seen above, LM-entailment does not satisfy all the proposed properties. An alternative definition, proposed by Booth and Paris (1998) is not applicable here since, in the language of PTL, we are not guaranteed that the final interpretation we obtain is still a ranked model of the relevant KB. We shall therefore consider a third option, one that is directly derived from the characterisation of rational closure by Giordano et al. (2012). The general idea is to respect the *presumption of typicality* ((Lehmann 1995), p.63). Such a principle indicates the way in which the property (RM) should be satisfied: we have  $\alpha \sim \gamma$  in our KB  $\mathcal{K}$ ; in order to satisfy (RM) we have to add either  $\alpha \sim \neg\beta$  or  $\alpha \wedge \beta \sim \gamma$ ; the presumption of typicality imposes that, whenever it is possible, we prefer the latter (that corresponds to a constrained application of monotony) over the former. Semantically, given the ranked models of a KB  $\mathcal{K}$ , such a presumption corresponds to considering only those interpretations in which every valuation is considered as typical as possible, that is, it is ‘pushed downward’ in the interpretation as much as possible, modulo the satisfaction of  $\mathcal{K}$ .

In order to identify the interpretations that can be interesting for the definition of a notion of entailment, we can introduce a preference relation  $\leq$  between the ranked interpretations that follows directly from the presumption of typicality. To do that, we need a notion to compare the relative positions of the valuations between the models of a KB.

**Definition 6 (Height function)** For a (finite) ranked interpretation  $\mathcal{R} = \langle \mathcal{V}, < \rangle$  and  $v \in \mathcal{V}$ , the height  $h_{\mathcal{R}}(v)$  of  $v$  corresponds to the number of the layer in  $\mathcal{R}$  in which  $v$  is positioned, or to  $\infty$  if it is not in the interpretation. That is, given that  $\mathcal{R} = (L_0, \dots, L_n)$ ,

$$h_{\mathcal{R}}(v) = \begin{cases} i & \text{if } v \in L_i, \text{ for } 0 \leq i \leq n \\ \infty & \text{otherwise.} \end{cases}$$

The lower the height of a valuation in an interpretation, the more typical such a valuation is considered in the ranked interpretation, while the height value is  $\infty$  if the valuation does not appear in the interpretation at all. Using the notion of height we can define a preorder over the interpretations.

**Definition 7 (Relation  $\leq$ )** Given a knowledge base  $\mathcal{K}$  and two of its models  $\mathcal{R} = \langle \mathcal{V}_{\mathcal{R}}, <_{\mathcal{R}} \rangle$  and  $\mathcal{R}' = \langle \mathcal{V}_{\mathcal{R}'}, <_{\mathcal{R}'} \rangle$ ,  $\mathcal{R}$  is at least as preferred as  $\mathcal{R}'$  ( $\mathcal{R} \leq \mathcal{R}'$ ) iff for every  $w \in \mathcal{U}$ ,  $h_{\mathcal{R}}(w) \leq h_{\mathcal{R}'}(w)$ .  $\mathcal{R}$  is preferred to  $\mathcal{R}'$  ( $\mathcal{R} < \mathcal{R}'$ ) iff  $\mathcal{R} \leq \mathcal{R}'$  and  $\mathcal{R}' \not\leq \mathcal{R}$ .

Consistent with the use of the *presumption of typicality* as a guideline in the choice of the relevant interpretations, the relation  $\leq$  can be used to identify the relevant interpretations for the definition of a notion of entailment: that is, we should choose the models of  $\mathcal{K}$  in which the valuations are presumed to be as typical as possible, that is, the relevant models are those that are in  $\min_{\leq}(\text{Mod}(\mathcal{K}))$ , with

$$\min_{\leq}(\text{Mod}(\mathcal{K})) = \{\mathcal{R} \in \text{Mod}(\mathcal{K}) \mid \nexists \mathcal{R}' \text{ s.t. } \mathcal{R}' \in \text{Mod}(\mathcal{K}) \text{ and } \mathcal{R}' < \mathcal{R}\}.$$

The use of the value  $\infty$  for the height function allows us to compare also interpretations containing distinct sets of valuations. Such a possibility is desirable since we do not want to consider as a possible candidate for a notion of entailment an interpretation that is a sub-interpretation of another one: given the models of a knowledge base and considering the minimal ones, we prefer those that do not make unjustified assumptions w.r.t. the classic knowledge, that is, we prefer the models that do not eliminate valuations without having the presumption of typicality as motivation.

It is easy to check that  $\leq$  is a preorder. Now, given a knowledge base  $\mathcal{K}$ , to define a closure operation we consider only the (preferred) models contained in  $\min_{\leq}(\text{Mod}(\mathcal{K}))$ .

If we are using KBs composed only of the classical non-monotonic conditionals  $\alpha \sim \beta$ , it corresponds exactly to LM-minimality as defined in the previous sections (Giordano et al. 2012). However, due to the expressivity of our language we obtain the surprising result that the two semantic constructions are not equivalent anymore. Moreover, in the present context, this notion of minimality can give back a number of minimal models, as the following example shows.

**Example 3** Consider the knowledge base  $\mathcal{K}$  from Example 2. The interpretations in  $\min_{\leq}(\text{Mod}(\mathcal{K}))$  are the following ones:

$\mathcal{R}_1$	$L_0 : \quad \{\neg p, \neg q, t\} \quad \{\neg p, \neg q, \neg t\}$
$\mathcal{R}_2$	$L_2 : \quad \{p, \neg q, \neg t\}$ $L_1 : \quad \{\neg p, \neg q, t\} \quad \{p, \neg q, t\}$ $L_0 : \quad \{\neg p, \neg q, \neg t\}$
$\mathcal{R}_3$	$L_2 : \quad \{\neg p, q, t\}$ $L_1 : \quad \{\neg p, q, \neg t\} \quad \{\neg p, \neg q, \neg t\}$ $L_0 : \quad \{\neg p, \neg q, t\}$

Note that the first of the three models in the example is the LM-minimal model of  $\mathcal{K}$ . In fact it is easy to check from the characterisation of rational closure in Section 3 and Definition 7 that every LM-minimal model of a KB  $\mathcal{K}$  is also in  $\min_{\leq}(\text{Mod}(\mathcal{K}))$ .

**Proposition 5** For every knowledge base  $\mathcal{K}$ , the LM-minimal model of  $\mathcal{K}$  is in  $\min_{\leq}(\text{Mod}(\mathcal{K}))$ .

We can define the notion of entailment  $\models_{\leq}$  corresponding to the present notion of minimality, as

$$\mathcal{K} \models_{\leq} \alpha \text{ iff } \forall \mathcal{R} \in \min_{\leq}(\text{Mod}(\mathcal{K})), \mathcal{R} \Vdash \alpha$$

This notion is inferentially weaker than the one based on LM-minimality since it is defined on a possibly larger set of models. Can this notion of entailment satisfy all our desiderata? It satisfies most of them.

**Proposition 6**  $\models_{\leq}$  satisfies (P1), (P2), (P3), (P4),(P7),(P8), and (P9).

Unfortunately, *Conditional Rationality* (P5) is not valid.

**Proposition 7** The conditional  $\sim$  induced by  $\models_{\leq}$  is not a rational conditional.

To see this, consider Example 3: we have that  $\mathcal{K} \models_{\leq} \neg p \sim \neg q$ , but we do not have neither  $\mathcal{K} \models_{\leq} \neg p \sim t$ , nor  $\mathcal{K} \models_{\leq} \neg p \wedge \neg t \sim \neg q$ .

We can consider another candidate for a notion of entailment, in order to augment the inferential power w.r.t. ranked entailment and preserve also Strict Entailment. We could consider only the models in  $\min_{\leq}(\mathcal{K})$  with the *biggest* (w.r.t.  $\subseteq$ ) universe. That is we would consider a set  $\min_{\subseteq}(\mathcal{K})$  s.t.

$$\min_{\subseteq}(\mathcal{K}) = \{\mathcal{R} = (\mathcal{V}, <) \mid \mathcal{R} \in \min_{\leq}(\mathcal{K}) \text{ and } \nexists \mathcal{R}' = (\mathcal{V}', <') \text{ s.t. } \mathcal{R}' \in \min_{\leq}(\mathcal{K}) \text{ and } \mathcal{V}' \supset \mathcal{V}\}$$

And the corresponding entailment relation  $\models_{\subseteq}$  would be

$$\mathcal{K} \models_{\subseteq} \alpha \text{ iff } \forall \mathcal{R} \in \min_{\subseteq}(\mathcal{K}), \mathcal{R} \Vdash \alpha$$

For instance, in Example 3 we would consider only  $\mathcal{R}_2$  and  $\mathcal{R}_3$ .  $\models_{\subseteq}$  is inferentially stronger than  $\models_{\leq}$ .

**Proposition 8**  $\models_{\leq} \subset \models_{\subseteq}$ .

But using  $\models_{\subseteq}$  we obtain an entailment relation that once again does not satisfy (P5), and moreover does not satisfy (P9).

**Proposition 9**  $\models_{\subseteq}$  does not satisfy Conditional Rationality (P5) and Typical Entailment (P9).

Both Proposition 8 and the failure of (P9) can be checked looking at Example 3:  $\mathcal{K} \models_{\subseteq} (\bullet \top \rightarrow t) \vee (\bullet \top \rightarrow \neg t)$ , that does not hold considering  $\models_{\leq}$  (and obviously it does not hold also for ranked entailment). For (P5) consider the same example used for  $\models_{\leq}$ .

So, we are not able to find an entailment relation for our enriched language that satisfies all our desiderata. Actually, we can give the following impossibility result, showing that some postulates stand in conflict with some of the others.

**Proposition 10** There is no consequence operator  $Cn_{\sim}(\cdot)$  satisfying all of Inclusion (P1), Single Model (P6), Strict Entailment (P8) and Typical Entailment (P9).

## 7 Concluding remarks

The main contributions of the present work can be summarized as follows: (i) the provision of a set of desiderata characterizing the behavior of a notion of entailment deemed appropriate in the context of PTL; (ii) the definition of the notion of minimum entailment and its assessment against the stated desiderata—in particular we provide a constructive method for generating the (finite) minimum model; (iii) an

investigation of alternative definitions of minimal entailment and their relationship with our constructions and desiderata.

Our results in the propositional setting pave the way for an investigation of appropriate forms of entailment in other, more expressive, preferential approaches, such as preferential description logics (Britz, Meyer, and Varzinczak 2011; Britz et al. 2013; Giordano et al. 2013) and logics of defeasible modalities (Britz and Varzinczak 2013). The move to logics with more structure, both in the syntax and in the semantics, is of a challenging nature, and a simple rephrasing of our approach to these logics may not deliver the expected results. We are currently investigating these issues.

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